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ORIGINAL PAPER

The architecture of polyhedral links and their HOMFLY polynomials

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Abstract A general approach is proposed to elucidate the topological characteristics of molecules with the shape of polyhedral links. For an arbitrary polyhedral graph, four classes of polyhedral links can be obtained by applying the operation of '*X-tangle covering*' to the related reduced sets. The relationships between the W-polynomial of a polyhedral graph and the HOMFLY polynomial of each kind of polyhedral links are established. These relationships not only simplify the computation but also provide a method of constructing a general formula for the HOMFLY polynomial of polyhedral links.

Keywords Polyhedral links · HOMFLY polynomial · W-polynomial · DNA Polyhedra

1 Introduction

Knot theory [1,2], the study of configurations of graphs in space, is a hot subject which is being rapidly developed, based on the remarkable progress of topology. It is now clear that knots and links are new forms of the molecular structures [3]. Since the first topological catenane [4] was synthesized in 1961, exciting advances have been made in the discoveries and syntheses of molecular knots [5], molecular catenanes [6,7], DNA knots and DNA catenanes [8–11], especially in the controls and syntheses of

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DNA polyhedral links or catenanes [12,13], such as the DNA tetrahedron [14–16], DNA cube [17,18], DNA truncated octahedron [19], DNA octahedron [20,21], DNA dodecahedron [16,22], DNA icosahedron [23,24], DNA bipyramid [25], and DNA buckyballs [16]. These peculiar objects provide some topological nontrivial structures embedding in 3D space, and to characterize them using mathematical methods is extremely interesting and challenging.

During the past few years, geometrical characterizations of the synthesized molecules with polyhedral shapes have attracted a lot of interest [26–31]. Qiu et al. fabricated Goldberg polyhedral and carbon nanotube links by the means of 'three cross curves and double lines covering' [26,27]. Subsequently, other construction methods have been developed for various links [28–31]. These studies of polyhedral links provide a sound basis of studying, designing and creating more topologically nontrivial molecule structures. It aroused our interest to try other methods in the architecture of polyhedral links. In the present work, we apply the operation of '*X-tangle covering*' to construct polyhedral links which can be widely used to characterize DNA polyhedral links or catenanes from the viewpoint of geometry and topology.

The HOMFLY polynomial [32], as an invariant of oriented links, is of significant importance because of its capabilities in identifying topological characteristics such as the topological link type and chirality of links [33]. In the aspect of calculating the HOMFLY polynomial for links, sometimes it is difficult, particularly for polyhedral links. Therefore, it is necessary to develop a simple method that can simplify the calculations. To do this, the W-polynomial [34,35] was introduced, and its connection with the HOMFLY polynomial was established. We expect the computation of the HOMFLY polynomial for polyhedral links to be simplified, and thus facilitating the subsequent identification of the topological link type and chirality of polyhedral links.

2 Construction of polyhedral links

First, some basic concepts, terminology and notation are stated. Meanwhile, two important operations for the construction of polyhedral links are given.

In graph theory, a planar graph is a graph which can be embedded in the plane or the sphere. A planar graph already drawn in the plane without edge intersections is called a plane graph. All convex polyhedrons are 3-connected planar graph [36], hence any one of them has a plane graph at least. While in this paper, 'polyhedral graph' means one of its plane graphs.

A signed graph is a graph in which every edge is designated to be either positive (+) or negative (−). If a signed graph is also a plane graph, it is called a signed plane graph. To obtain alternating links, the signed graphs involved in the paper are assumed to have all signs of the same type. In addition, when each edge of a graph G is assigned a positive sign (+), the graph can be denoted as G^+. On the contrary, the graph G^- means each edge of a graph G being assigned a negative sign (−). Note that G^+ is the mirror image of G^-.

A reduced graph is a graph in which the multiple edges are deleted, and the edges that are adjacent to the vertex of degree 2 are contracted. A series of graphs with the same reduced graph form a reduced set, and it is denoted as $R(G)$. For a signed graph,

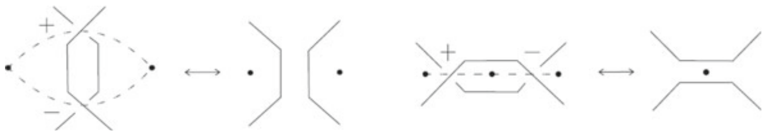

Fig. 1 Reidemeister move II (*Dotted line* denotes plane graph, *solid line* denotes the corresponding link diagram)

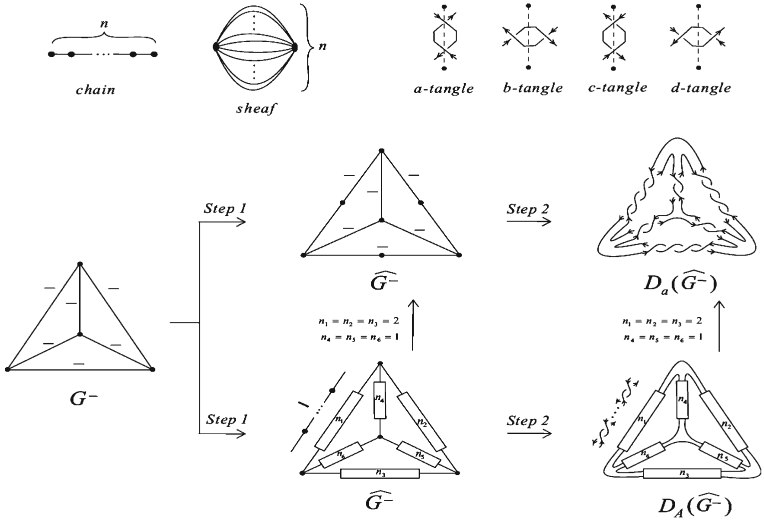

Fig. 2 The construction of $D_a(\widehat{G^-})$ and $D_A(\widehat{G^-})$ (First, we choose a chain to replace each edge of G^-, where each box contains a chain of length n_i with the sign $-$; second, each box contains n_i a-tangle in $D_A(\widehat{G^-})(i = 1 \ldots 6.)$

its reduced graph is usually the reduced graph of the underlying unsigned graph. For example, in Fig. 2, G^- is the reduced graph of $\widehat{G^-}$. Noticeably, convex polyhedrons do not have multiple edges and the vertex of degree 2, so their reduced graphs are themselves.

A chain is a graph which is a path. A sheaf is a graph with two vertices connected by some parallel edges. We define the length of a path (respectively the width of a sheaf) as the number of edges in the path (respectively the sheaf).

Given a signed plane graph G, we replace each edge with a chain or sheaf to obtain a new signed plane graph. This operation is called '*chain or sheaf replacing*', as illustrated in Fig. 2. The new graph is denoted as $\widehat{G}$. Note that if two adjacent edges have different type of signs on a chain or a sheaf, we can offset the edges by Reidemeister move II (see Fig. 1). Therefore, we assume that each edge of a chain or a sheaf has the same type of sign with the edge replaced.

Given a signed plane graph G, we use a X-tangle to cover each edge, and connect the adjacent ends of two tangles if the related edges are adjacent in a face of G, which result in a link diagram, denoted by $D_X(G)$. This operation is called '*X-tangle covering*' where X may be a, b, c or d (see Fig. 2). It extends the Jeager link [37], and

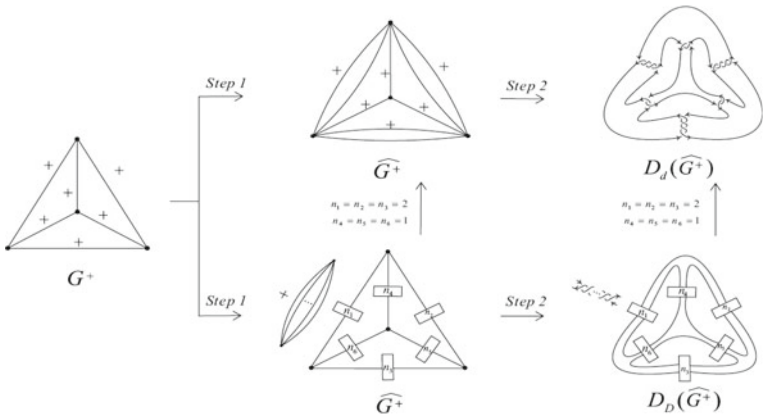

Fig. 3 The construction of $D_d(\widehat{G^+})$ and $D_D(\widehat{G^+})$ (First, we choose a sheaf to replace each edge of G^+, where each box contains a sheaf of width n_i with the sign +; second, each box contains n_i d-tangle in $D_D(\widehat{G^+})(i = 1\ldots6)$.)

is similar to the idea of Traldi [35], hence it does not cause conflict for the orientation of the polyhedral link obtained by applying '*X-tangle covering*' to a polyhedral graph.

Given a polyhedral graph G, it has two signed graph G^- and G^+, and their reduced sets are $R(G^-)$ and $R(G^+)$, respectively. The construction of polyhedral links is based on the knowledge of topology and graph theory.

2.1 Four polyhedral links

For G^- (or G^+), applying the operation of '*chain or sheaf replacing*' results in a new signed graph $\widehat{G^-}$ (or $\widehat{G^+}$). And then the applications of the operations of '*a-tangle covering*' and '*b-tangle covering*' on $\widehat{G^-}$ result in two related link diagrams $D_a(\widehat{G^-})$ and $D_b(\widehat{G^-})$, respectively. Similarly, the operations of '*c-tangle covering*' and '*d-tangle covering*' on $\widehat{G^+}$ lead to the two related link diagrams $D_c(\widehat{G^+})$ and $D_d(\widehat{G^+})$, respectively. In fact, by changing chain length and sheaf width, we can obtain each graph of $R(G^-)$ (or $R(G^+)$) from $\widehat{G^-}$ (or $\widehat{G^+}$).

2.2 Four classes of polyhedral links

The operation of '*a-tangle covering*' (or '*b-tangle covering*') on each graph of $R(G^-)$ results in a class of polyhedral links which is denoted as $D_A(\widehat{G^-})$(or $D_B(\widehat{G^-})$).

Similarly, the operation of '*c-tangle covering*' (or '*d-tangle covering*') on each graph of $R(G^+)$ leads to a class of polyhedral links which is denoted as $D_D(\widehat{G^+})$ (or $D_C(\widehat{G^+})$). We give an example of the operations on tetrahedron, as shown in Figs. 2 and 3.

3 The relationship between the W-polynomial and the HOMFLY polynomial

First of all, some terminologies are introduced.

An isthmus of a graph is an edge whose deletion increases the number of components (i.e. disconnects the graph if it was originally connected). A loop of a graph is an edge whose endpoints are the same. Hereinafter we use $G - e$ and $G \cdot e$ to denote the graphs obtained from graph G by deleting and contracting edge e. $|V(G)|$ is the number of vertexes of a graph G, and $|E(G)|$ is the number of edges. Here, $X = A + Bd$ and $Y = Ad + B$.

3.1 The Tutte polynomial and the HOMFLY polynomial

The relationships between the Tutte polynomials [38,39] of signed graphs and the HOMFLY polynomials of four related links, as a special case of Traldi's results [35], can be used to understand the connection between signed graph and four related links obtained in the Sect. 2.

Definition 3.1 The Tutte polynomial $T(G, x, y) \in \mathbb{Z}[x, y]$ for a graph G is defined by the following recursion formulas:

(1) Suppose that G only has an edge e. If e is an isthmus (*respectively* loop), then

$$T(G, x, y) = x(respectively\, T(G, x, y) = y).$$

(2) If edge e is neither an isthmus nor loop, then

$$T(G, x, y) = T(G - e, x, y) + T(G \cdot e, x, y).$$

(3) Suppose that G has two edges or more. If edge e is an isthmus (*respectively* loop), then

$$T(G, x, y) = xT(G \cdot e, x, y)(\text{respectively}\, T(G, x, y) = yT(G - e, x, y)).$$

Definition 3.2 The HOMFLY polynomial $H(L, x, y, z) \in \mathbb{Z}[x, y, z]$ for an oriented link L is defined by the following relationships:

(1) If L is a trivial knot, then

$$H(L, x, y, z) = 1.$$

(2) If two links L_1 and L_2 are equivalent under ambient isotopic, then

$$H(L_1, x, y, z) = H(L_2, x, y, z).$$

(3) Suppose that three link diagrams L_+, L_- and L_0 are different only on a local region, as shown in Fig. 4, then

$$xH(L_+, x, y, z) + yH(L_-, x, y, z) + zH(L_0, x, y, z) = 0.$$

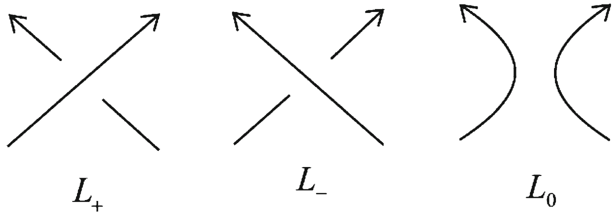

Fig. 4 Three link diagrams: L_+, L_- and L_0

Therefore, we can obtain the HOMFLY polynomial in two variables:

$$H(L, v, z) = H(L, v^{-1}, -v, -z).$$

The HOMFLY polynomial has the following properties:

(1) If L is the connected sum of L_1 and L_2, denoted by $L_1 \# L_2$, then

$$H(L, x, y, z) = H(L_1, x, y, z)H(L_2, x, y, z).$$

(2) If L is the disjoint union of L_1 and L_2, denoted by $L_1 \cup L_2$, then

$$H(L, x, y, z) = \left(-\frac{x+y}{z}\right) H(L_1, x, y, z)H(L_2, x, y, z).$$

(3) If L^* is the mirror image of L, then

$$H(L^*, v, z) = H(L, v^{-1}, z)$$

It shows that the HOMFLY polynomial of an achiral link must satisfy:

$$H(L, v, z) = H(L, v^{-1}, z).$$

Theorem 3.3 *Let $\widehat{G^-}$ be a connected signed plane graph. Let $D_a(\widehat{G^-})$ be the link obtained by applying the operation of ' a-tangle covering' to $\widehat{G^-}$. Then*

$$H(D_a(\widehat{G^-}), x, y, z) = \left(\frac{y}{z}\right)^{|V(\widehat{G^-})|-1} \left(-\frac{z}{x}\right)^{|E(\widehat{G^-})|} T\left(\widehat{G^-}, -\frac{x}{y}, 1 - \frac{xy + y^2}{z^2}\right).$$

Theorem 3.3, obtained by Jeager [37], establishes the connection between plane graph and its related link. Here, we establish the relationship between signed graphs and the other three related links.

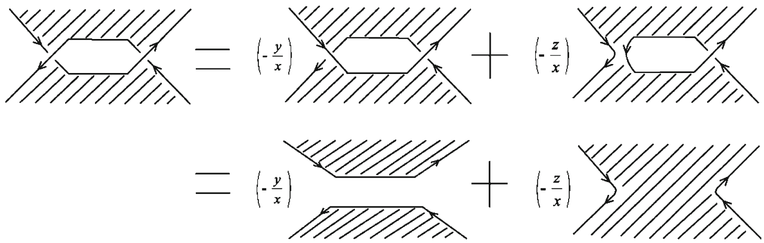

Fig. 5 Five diagrams which differ only on a local region. (Each diagram L stands for the value of *H(L, x, y, z)*)

Theorem 3.4 *Let $\widehat{G^+}$ be a connected signed plane graph. Let $D_d(\widehat{G^+})$ be the link obtained by applying the operation of 'd-tangle covering' to $\widehat{G^+}$. Then*

$$H(D_d(\widehat{G^+}), x, y, z) = \left(\frac{z}{y}\right)^{|V(\widehat{G^+})|-1} \left(-\frac{y}{x}\right)^{|E(\widehat{G^+})|} T\left(\widehat{G^+}, 1 - \frac{xy + y^2}{z^2}, -\frac{x}{y}\right). \tag{1}$$

Proof We use induction on the number of edges of $\widehat{G^+}$.

Let e be an edge of $\widehat{G^+}$. We apply the Definition (3) of the HOMFLY polynomial to the crossing of $D_d(\widehat{G^+})$ associated to e, which results in the equation depicted pictorially in Fig. 5.

(1) Suppose that $|E(\widehat{G^+})| = 1$.

If e is a loop, then $H(D_d(\widehat{G^+}), x, y, z) = 1$ and $T\left(\widehat{G^+}, 1 - \frac{xy+y^2}{z^2}, -\frac{x}{y}\right) = -\frac{x}{y}$. If e is an isthmus, then

$$H(D_d(\widehat{G^+}), x, y, z) = \left(-\frac{z}{x}\right)\left(1 - \frac{xy + y^2}{z^2}\right) \quad \text{and}$$
$$T\left(\widehat{G^+}, 1 - \frac{xy + y^2}{z^2}, -\frac{x}{y}\right) = 1 - \frac{xy + y^2}{z^2}.$$

The formula (1) can be easily checked by using the above results.

(2) Suppose that $|E(\widehat{G^+})| > 1$.

i) If e is neither a loop nor an isthmus, then

$$H\left(D_d(\widehat{G^+}), x, y, z\right) = \left(-\frac{y}{x}\right) H(D_d\left(\widehat{G^+} - e), x, y, z\right) + \left(-\frac{z}{x}\right) H\left(D_d(\widehat{G^+} \cdot e), x, y, z\right).$$

Because $\widehat{G^+} - e$ and $\widehat{G^+} \cdot e$ are connected, we may apply our induction hypothesis to obtain

$$H\left(D_d(\widehat{G^+} - e), x, y, z\right) = \left(\frac{z}{y}\right)^{|V(\widehat{G^+})|-1} \left(-\frac{y}{x}\right)^{|E(\widehat{G^+})|-1} T\left(\widehat{G^+} - e, 1 - \frac{xy + y^2}{z^2}, -\frac{x}{y}\right)$$

and

$$H\left(D_d(\widehat{G^+} \cdot e), x, y, z\right) = \left(\frac{z}{y}\right)^{|V(\widehat{G^+})|-2} \left(-\frac{y}{x}\right)^{|E(\widehat{G^+})|-1} T\left(\widehat{G^+} \cdot e, 1 - \frac{xy + y^2}{z^2}, -\frac{x}{y}\right).$$

Consequently,

$$\begin{aligned} H(D_d(\widehat{G^+}), x, y, z) = {} & \left(-\frac{y}{x}\right)\left(\frac{z}{y}\right)^{|V(\widehat{G^+})|-1} \left(-\frac{y}{x}\right)^{|E(\widehat{G^+})|-1} \\ & T\left(\widehat{G^+} - e, 1 - \frac{xy + y^2}{z^2}, -\frac{x}{y}\right) \\ & + \left(-\frac{z}{x}\right)\left(\frac{z}{y}\right)^{|V(\widehat{G^+})|-2} \left(-\frac{y}{x}\right)^{|E(\widehat{G^+})|-1} \\ & T\left(\widehat{G^+} \cdot e, 1 - \frac{xy + y^2}{z^2}, -\frac{x}{y}\right). \end{aligned}$$

Hence, we can obtain the formula (1) from the above result.

ii) If e is a loop, it is evident that $D_d(\widehat{G^+})$ and $D_d(\widehat{G^+} - e)$ are ambient isotopic. Hence

$$H(D_d(\widehat{G^+}), x, y, z) = H(D_d(\widehat{G^+} - e), x, y, z) = \left(\frac{z}{y}\right)^{|V(\widehat{G^+})|-1} \left(-\frac{y}{x}\right)^{|E(\widehat{G^+})|-1} T\left(\widehat{G^+} - e, 1 - \frac{xy + y^2}{z^2}, -\frac{x}{y}\right).$$

Using the Definition (3) of the Tutte polynomial, we can obtain the required result.

iiii) If e is an isthmus, then deleting e will result in the disjoint union of two links, it is denoted by $L_1 \cup L_2$. While contracting e will result in the graph, it is expressed as $L_1 \# L_2$. They satisfy the following equation

$$H(D_d(\widehat{G^+}), x, y, z) = \left(-\frac{y}{x}\right) H(L_1 \cup L_2, x, y, z) + \left(-\frac{z}{x}\right) H(L_1 \# L_2, x, y, z).$$

From the properties (1) and (2) of the HOMFLY polynomial, it can realize that

$$H(L_1 \cup L_2, x, y, z) = \left(-\frac{x+y}{z}\right) H(L_1 \# L_2, x, y, z).$$

Hence

$$H(D_d(\widehat{G^+}), x, y, z) = \left(\frac{xy + y^2 - z^2}{xz}\right) H(L_1 \# L_2, x, y, z).$$

Because $L_1 \# L_2$ and $D_d(\widehat{G^+} \cdot e)$ are isotopic, thus

$$H(D_d(\widehat{G^+}), x, y, z) = \left(\frac{xy + y^2 - z^2}{xz}\right) H(D_d(\widehat{G^+} \cdot e), x, y, z).$$

Because $\widehat{G^+} \cdot e$ is connected and $|E(\widehat{G^+} \cdot e)| = |E(\widehat{G^+})| - 1$, we may apply our induction hypothesis to obtain

$$H(D_d(\widehat{G^+} \cdot e), x, y, z) = \left(\frac{z}{y}\right)^{|V(\widehat{G^+})|-2} \left(-\frac{y}{x}\right)^{|E(\widehat{G^+})|-1} T\left(\widehat{G^+} \cdot e, 1 - \frac{xy + y^2}{z^2}, -\frac{x}{y}\right)$$

Hence

$$H(D_d(\widehat{G^+}), x, y, z) = \left(1 - \frac{xy + y^2}{z^2}\right) \left(\frac{z}{y}\right)^{|V(\widehat{G^+})|-1} \left(-\frac{y}{x}\right)^{|E(\widehat{G^+})|} T\left(\widehat{G^+} \cdot e, 1 - \frac{xy + y^2}{z^2}, -\frac{x}{y}\right).$$

Using the Definition (3) of the Tutte polynomial, we can obtain the formula (1). □

Since $D_a(\widehat{G^-})$ (resp. $D_b(\widehat{G^-})$) is the mirror image of $D_c(\widehat{G^+})$ (resp. $D_d(\widehat{G^+})$), we can give the following theorems.

Theorem 3.5 *Let $\widehat{G^-}$ be a connected signed plane graph. Let $D_b(\widehat{G^-})$ be the link obtained by applying the operation of 'b-tangle covering' to $\widehat{G^-}$. Then*

$$H(D_b(\widehat{G^-}), x, y, z) = \left(\frac{z}{x}\right)^{|V(\widehat{G^-})|-1} \left(-\frac{x}{y}\right)^{|E(\widehat{G^-})|} T\left(\widehat{G^-}, 1 - \frac{xy + x^2}{z^2}, -\frac{y}{x}\right).$$

Theorem 3.6 *Let $\widehat{G^+}$ be a connected signed plane graph. Let $D_c(\widehat{G^+})$ be the link obtained by applying the operation of 'c-tangle covering' to $\widehat{G^+}$. Then*

$$H(D_c(\widehat{G^+}), x, y, z) = \left(\frac{x}{z}\right)^{|V(\widehat{G^+})|-1} \left(-\frac{z}{y}\right)^{|E(\widehat{G^+})|} T\left(\widehat{G^+}, -\frac{y}{x}, 1 - \frac{xy + x^2}{z^2}\right).$$

3.2 The Q-polynomial and the HOMFLY polynomial

In this section, the connection between the Q-polynomial and the Tutte polynomial is established by using Dichromatic polynomial. Based on their connection, the relationships between the Q-polynomials of the signed graphs and the HOMFLY polynomials of the related links obtained by using the construction method in Sect. 2 are established.

Definition 3.7 The Dichromatic polynomial $Z[G](q, v) \in \mathbb{Z}[x, y]$ for a graph G is defined by the following recursive formulas:

(1) If G is composed of an isolated vertex, then $Z[\cdot] = q$.
(2) Let e be an edge of graph G. Then

$$Z[G] = Z[G - e] + vZ[G \cdot e].$$

(3) Let $G \cup H$ be the disjoint union of the graphs G and H, we have

$$Z[G \cup H] = Z[G]Z[H].$$

The Dichromatic polynomial and the Tutte polynomial are equivalent, and they are connected by the following formula:

$$T(G, x, y) = (x - 1)^{-1}(y - 1)^{-|V(G)|}Z[G]((x - 1)(y - 1), y - 1).$$

Definition 3.8 The Q-polynomial $Q[G] = Q[G](A, B, d) \in \mathbb{Z}[x, y, z]$ for a signed graph G is defined by the following recursive formulas:

(1) Suppose that e is neither an isthmus nor a loop in G. Then
   i) $Q[G] = AQ[G - e] + BQ[G \cdot e]$ sign $(e) = -$,
   ii) $Q[G] = BQ[G - e] + AQ[G \cdot e]$ sign $(e) = +$.
(2) Suppose that G is connected and consists entirely of isthmus and loops. Here we use i_- and i_+ to denote the number of negative and positive isthmuses, l_- and l_+ the number of negative and positive loops. Then

$$Q[G] = X^{i_+ + l_-}Y^{i_- + l_+}.$$

(3) Suppose G is the disjoint union of the graphs G_1 and G_2. Then

$$Q[G] = dQ[G_1]Q[G_2].$$

Lemma 3.9 [40] *We use $Z[G^+](q, v)$ to denote the Dichromatic polynomial of the underlying unsigned graph. Then*

$$Z[G^+](q, v) = q^{\frac{|V(G^+)|+1}{2}} Q[G^+]\left(q^{-\frac{1}{2}}v, 1, q^{\frac{1}{2}}\right).$$

Similarly, the following lemma is given as follows.

Lemma 3.10 *We use $Z[G^-](q, v)$ to denote the Dichromatic polynomial of the underlying unsigned graph. Then*

$$Z[G^-](q, v) = q^{\frac{|V(G^-)|+1}{2}} Q[G^-]\left(1, q^{-\frac{1}{2}}v, q^{\frac{1}{2}}\right).$$

Lemma 3.11 *Let G be a signed graph. If $G = G^+$, then*

$$T(G, x, y) = (x-1)^{\frac{|V(G)|-1}{2}}(y-1)^{-\frac{|V(G)|-1}{2}} Q[G]((x-1)^{-\frac{1}{2}} \times (y-1)^{\frac{1}{2}}, 1, (x-1)^{\frac{1}{2}}(y-1)^{\frac{1}{2}}).$$

If $G = G^-$, then

$$T(G, x, y) = (x-1)^{\frac{|V(G)|-1}{2}}(y-1)^{-\frac{|V(G)|-1}{2}} Q[G](1, (x-1)^{-\frac{1}{2}} \times (y-1)^{\frac{1}{2}}, (x-1)^{\frac{1}{2}}(y-1)^{\frac{1}{2}}).$$

Lemma 3.11 can be deduced from Lemmas 3.9 and 3.10, and the relationship between the Dichromatic polynomial and the Tutte polynomial. The proof is omitted here.

Theorem 3.12 *Let $\widehat{G^-}$ and $\widehat{G^+}$ be two signed plane graphs. Let $D_a(\widehat{G^-})$, $D_b(\widehat{G^-})$, $D_c(\widehat{G^+})$ and $D_d(\widehat{G^+})$ be four oriented links obtained by using the construction method in Sect. 2. The relationships between the graphs and the links are as follows:*

(1) $H(D_a(\widehat{G^-}), x, y, z) = \left(-\frac{z}{x}\right)^{|E(\widehat{G^-})|} Q[\widehat{G^-}]\left(1, \frac{y}{z}, -\frac{x+y}{z}\right)$,

(2) $H(D_b(\widehat{G^-}), x, y, z) = \left(-\frac{x}{y}\right)^{|E(\widehat{G^-})|} Q[\widehat{G^-}]\left(1, \frac{z}{x}, -\frac{x+y}{z}\right)$,

(3) $H(D_c(\widehat{G^+}), x, y, z) = \left(-\frac{z}{y}\right)^{|E(\widehat{G^+})|} Q[\widehat{G^+}]\left(\frac{x}{z}, 1, -\frac{x+y}{z}\right)$,

(4) $H(D_d(\widehat{G^+}), x, y, z) = \left(-\frac{y}{x}\right)^{|E(\widehat{G^+})|} Q[\widehat{G^+}]\left(\frac{z}{y}, 1, -\frac{x+y}{z}\right)$.

Proof Here, we take the formula (1) as an example and prove it. The proof for other formulas is similar. By means of Theorem 3.3 and Lemma 3.11, we have

$$H(D_a(\widehat{G^-}), x, y, z) = C Q[\widehat{G^-}](A, B, d),$$

where $A = 1$, $B = \pm\frac{y}{z}$, $d = \pm\frac{x+y}{z}$ and $C = \pm\left(-\frac{z}{x}\right)^{|E(\widehat{G^-})|}$.

We discuss several cases for the determination of B, d, C.
Suppose that $\widehat{G^-}$ has only an edge e. Then $|E(\widehat{G^-})| = 1$.

i) If e is an isthmus, then

$$Q[\widehat{G^-}] = Ad + B \text{ and } H(D_a(\widehat{G^-}), x, y, z) = 1$$

(1) If $B = \frac{y}{z}$ and $d = \frac{x+y}{z}$, then

$$H(D_a(\widehat{G^-}), x, y, z) = \left(\pm\frac{z}{x}\right)\left(\frac{x+2y}{z}\right) \neq 1.$$

Hence, we have

$$B \neq \frac{y}{z} \quad \text{and} \quad d \neq \frac{x+y}{z}.$$

(2) If $B = \frac{y}{z}$, $d = -\frac{x+y}{z}$ and $C = -\frac{z}{x}$, then

$$H(D_a(\widehat{G^-}), x, y, z) = \left(-\frac{z}{x}\right)\left(-\frac{x}{z}\right) = 1.$$

(3) If $B = -\frac{y}{z}$ and $d = -\frac{x+y}{z}$, then

$$H(D_a(\widehat{G^-}), x, y, z) = \left(\pm\frac{z}{x}\right)\left(-\frac{x+2y}{z}\right) \neq 1.$$

Hence, we have

$$B \neq -\frac{y}{z} \quad \text{and} \quad d \neq -\frac{x+y}{z}.$$

(4) If $B = -\frac{y}{z}$, $d = \frac{x+y}{z}$ and $C = \frac{z}{x}$, then

$$H(D_a(\widehat{G^-}), x, y, z) = \left(\frac{z}{x}\right)\left(\frac{x}{z}\right) = 1.$$

We only discuss the cases of (2) and (4) in the following.

ii) If e is a loop, then

$$Q[\widehat{G^-}] = A + Bd \quad \text{and} \quad H(D_a(\widehat{G^-}), x, y, z) = \left(-\frac{z}{x}\right)\left(1 - \frac{xy + y^2}{z^2}\right)$$

(2) If $B = \frac{y}{z}$, $d = -\frac{x+y}{z}$ and $C = -\frac{z}{x}$, then

$$H(D_a(\widehat{G^-}), x, y, z) = \left(-\frac{z}{x}\right)\left(1 - \frac{xy + y^2}{z^2}\right).$$

(4) If $B = -\frac{y}{z}$, $d = \frac{x+y}{z}$ and $C = \frac{z}{x}$, then

$$H(D_a(\widehat{G^-}), x, y, z) = \left(\frac{z}{x}\right)\left(1 - \frac{xy + y^2}{z^2}\right).$$

It is clear that $B = \frac{y}{z}$, $d = -\frac{x+y}{z}$ and $C = -\frac{z}{x}$ can be obtained from the above discussion.

Hence, we can conclude that

$$H(D_a(\widehat{G^-}), x, y, z) = \left(-\frac{z}{x}\right)^{|E(\widehat{G^-})|} Q[\widehat{G^-}]\left(1, \frac{y}{x}, -\frac{x+y}{z}\right). \qquad \square$$

3.3 The HOMFLY polynomial and the W-polynomial

Based on the relationship between the Q-polynomial and the W-polynomial, together with the aforementioned relationship between the Q-polynomial and the HOMFLY polynomial, the relationship between the W-polynomial and the HOMFLY polynomial can be established. As a result, the connection between a plane graph and four classes of related links obtained in Sect. 2 can be derived.

We define a graph G as a colored graph, if there is a function f from edge set E to color set Λ.

Definition 3.13 The W-polynomial $W(G) = W(G)(t, z_1, z_2) \in \mathbb{Z}[x, y, z]$ for a colored graph G is defined by the following recursion formulas:

(1) Let E_n be a graph which is composed of n isolated vertexes, then

$$W(E_n) = t^{n-1}.$$

(2) We use c(e) to denote the color of edge e and assume that $c(e) = \lambda$.

  (a) When e is an isthmus,

$$W(G) = (x_\lambda + y_\lambda z_1)W(G \cdot e).$$

  (b) When e is a loop,

$$W(G) = (y_\lambda + x_\lambda z_2)W(G - e).$$

  (c) Other cases,

$$W(G) = x_\lambda W(G \cdot e) + y_\lambda W(G - e).$$

The Q-polynomial is generalized by the W-polynomial, and a signed graph G can be considered as a colored graph with $\Lambda = \{+, -\}$. The following theorem shows the interrelation between a signed graph G and the new signed graph $\widehat{G}$ obtained by applying the operation of '*chain or sheaf replacing*' to G.

Theorem 3.14 *Let G be a signed graph. Let $\widehat{G}$ be the graph obtained from G by replacing each edge with a chain or sheaf. For the edge replaced by length n of a chain, it will be colored with c_n^+ if the sign of the edge is positive; otherwise, it will be colored with c_n^-. For the edge replaced by width n of a sheaf, it will be colored with s_n^+ if the sign of the edge is positive; otherwise, it will be colored with s_n^-. In $W(G)$, if*

$$x_{c_n^+} = A^n,\ y_{c_n^+} = \frac{X^n - A^n}{d},\ x_{s_n^+} = \frac{Y^n - B^n}{d},\ y_{s_n^+} = B^n$$

and

$$x_{c_n^-} = B^n,\ y_{c_n^-} = \frac{Y^n - B^n}{d},\ x_{s_n^-} = \frac{X^n - A^n}{d},\ y_{s_n^-} = A^n,$$

then

$$Q(\widehat{G}) = W(G)(d, d, d).$$

This theorem can be directly obtained from Lemma 7 and Theorem 8 [41].

Theorem 3.15 *Let G be a plane graph. Let $D_A\widehat{(G^-)}, D_B\widehat{(G^-)}, D_C\widehat{(G^+)}$ and $D_D\widehat{(G^+)}$ be four classes of links obtained by using the construction method in Sect. 2.*

(1) In $W(G)$, if

$$x_{c_n^-} = \left(\frac{y}{z}\right)^n,\ y_{c_n^-} = \frac{\left(-\frac{x}{z}\right)^n - \left(\frac{y}{z}\right)^n}{\left(-\frac{x+y}{z}\right)},\ x_{s_n^-} = \frac{\left(1 - \frac{xy+y^2}{z^2}\right)^n - 1}{\left(-\frac{x+y}{z}\right)},\ y_{s_n^-} = 1,$$

then

$$H(D_A\widehat{(G^-)}, x, y, z) = \left(-\frac{z}{x}\right)^{|E\widehat{(G^-)}|} W(G)\left(-\frac{x+y}{z}, -\frac{x+y}{z}, -\frac{x+y}{z}\right).$$

(2) In $W(G)$, if

$$x_{c_n^-} = \left(\frac{z}{x}\right)^n,\ y_{c_n^-} = \frac{\left(-\frac{x+y}{z} + \frac{z}{x}\right)^n - \left(\frac{z}{x}\right)^n}{\left(-\frac{x+y}{z}\right)},\ x_{s_n^-} = \frac{\left(-\frac{y}{x}\right)^n - 1}{\left(-\frac{x+y}{z}\right)},\ y_{s_n^-} = 1,$$

then

$$H(D_B\widehat{(G^-)}, x, y, z) = \left(-\frac{x}{y}\right)^{|E\widehat{(G^-)}|} W(G)\left(-\frac{x+y}{z}, -\frac{x+y}{z}, -\frac{x+y}{z}\right).$$

(3) *In $W(G)$, if*

$$x_{c_n^+} = \left(\frac{x}{z}\right)^n, y_{c_n^+} = \frac{\left(-\frac{y}{z}\right)^n - \left(\frac{x}{z}\right)^n}{\left(-\frac{x+y}{z}\right)}, x_{s_n^+} = \frac{\left(1 - \frac{xy+x^2}{z^2}\right)^n - 1}{\left(-\frac{x+y}{z}\right)}, y_{s_n^+} = 1,$$

then

$$H(D_C(\widehat{G^+}), x, y, z) = \left(-\frac{z}{y}\right)^{|E(\widehat{G^+})|} W(G)\left(-\frac{x+y}{z}, -\frac{x+y}{z}, -\frac{x+y}{z}\right).$$

(4) *In $W(G)$, if*

$$x_{c_n^+} = \left(\frac{z}{y}\right)^n, y_{c_n^+} = \frac{\left(-\frac{x+y}{z} + \frac{z}{y}\right)^n - \left(\frac{z}{y}\right)^n}{\left(-\frac{x+y}{z}\right)}, x_{s_n^+} = \frac{\left(-\frac{x}{y}\right)^n - 1}{\left(-\frac{x+y}{z}\right)}, y_{s_n^+} = 1$$

then

$$H(D_D(\widehat{G^+}), x, y, z) = \left(-\frac{y}{x}\right)^{|E(\widehat{G^+})|} W(G)\left(-\frac{x+y}{z}, -\frac{x+y}{z}, -\frac{x+y}{z}\right).$$

This theorem completes our results. Its proof can be easily concluded from Theorems 3.12 and 3.14 (not stated here).

3.4 The application

The tetrahedron is taken as an example to explore topological properties of a class of polyhedral links. Let G be a tetrahedral graph. Let $\widehat{G^-}$ and $\widehat{G^+}$ be the graphs obtained by replacing each edge e with length n_e of a chain (see Fig. 2). As a result, the related four classes of polyhedral link $D_A(\widehat{G^-})$, $D_B(\widehat{G^-})$, $D_C(\widehat{G^+})$ and $D_D(\widehat{G^+})$ can be obtained by using the construction method in Sect. 2.

(1) The general formula of the HOMFLY polynomial of $D_A(\widehat{G^-})$. By using Theorem 3.15, we have the following result

$$H(D_A(\widehat{G^-}), x, y, z) = z^{-3}P_1 + z^{-1}P_2 + zP_3 + z^3P_4,$$

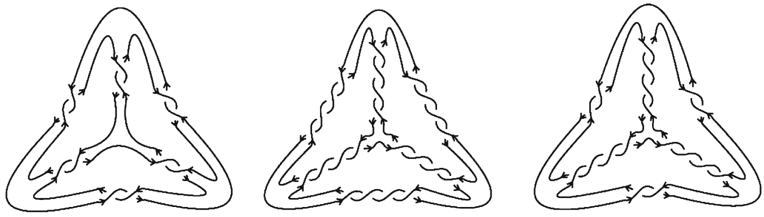

Fig. 6 Three tetrahedral link diagrams: L_1, L_2 and L_3

where

$$
\begin{aligned}
P_1 &= -v^{2N+3} + v^{2N-3} - 3v^{2N-1} + 3v^{2N+1},\\
P_2 &= 6v^{2N+1} - 6v^{2N-1}\\
&\quad - v^{2N+1-2n_6} - v^{2N+1-2n_5} - v^{2N+1-2n_4}\\
&\quad - v^{2N+1-2n_3} - v^{2N+1-2n_2} - v^{2N+1-2n_1} + v^{2N-1-2n_6}\\
&\quad + v^{2N-1-2n_5} + v^{2N-1-2n_4} + v^{2N-1-2n_3} + v^{2N-1-2n_2} + v^{2N-1-2n_1},\\
P_3 &= [11v^{2N} + v^{2N-2n_1-2n_6} + v^{2N-2n_2-2n_5}\\
&\quad + v^{2N-2n_3-2n_4} + v^{2n_1+2n_2+2n_3} + v^{2n_1+2n_4+2n_5}\\
&\quad + v^{2n_2+2n_4+2n_6} + v^{2n_3+2n_5+2n_6} - 3v^{2N-2n_6} - 3v^{2N-2n_5}\\
&\quad - 3v^{2N-2n_4} - 3v^{2N-2n_3} - 3v^{2N-2n_2} - 3v^{2N-2n_1}](v^{-1} - v)^{-1},\\
P_4 &= [1 - 6v^{2N} - v^{2N-2n_1-2n_6}\\
&\quad - v^{2N-2n_2-2n_5} - v^{2N-2n_3-2n_4} - v^{2n_1+2n_2+2n_3}\\
&\quad - v^{2n_1+2n_4+2n_5} - v^{2n_2+2n_4+2n_6} - v^{2n_3+2n_5+2n_6} + 2v^{2N-2n_6}\\
&\quad + 2v^{2N-2n_5} + 2v^{2N-2n_4} + v^{2N-2n_3} + v^{2N-2n_2} + v^{2N-2n_1}](v^{-1} - v)^{-3}\\
&\quad (N = n_1 + n_2 + n_3 + n_4 + n_5 + n_6).
\end{aligned}
$$

(2) For G, when $n_i = 1 \quad (i = 1 \cdots 6)$, we can get polyhedral link L_1 (see Fig. 6). Similarly, we can obtain polyhedral links L_2 and L_3 when $n_i = 2 \quad (i = 1 \cdots 6), n_1 = n_2 = n_3 = 1$ and $n_4 = n_5 = n_6 = 2$, respectively. According to the different value of n_i, we can obtain the HOMFLY polynomials of L_1, L_2 and L_3 from the general formula $H(D_A(\widehat{G^-}), v, z)$ as follows.

$$
\begin{aligned}
H(L_1, v, z) &= (v^9 - 3v^{11} + 3v^{13} - v^{15})z^{-3}\\
&\quad + (6v^9 - 12v^{11} + 6v^{13})z^{-1}\\
&\quad + (4v^7 + 7v^9 - 11v^{11})z\\
&\quad + (6v^9 + 6v^7 + 3v^5 + v^3)z^3,\\
H(L_2, v, z) &= (v^{21} - 3v^{23} + 3v^{25} - v^{27})z^{-3}\\
&\quad + (6v^{19} - 6v^{21} - 6v^{23} + 6v^{25})z^{-1}\\
&\quad + (4v^{13} + 4v^{15} + 7v^{17} + 7v^{19} - 11v^{21} - 11v^{23})z
\end{aligned}
$$

$$
\begin{aligned}
&+(v^3+3v^5+6v^7+10v^9+15v^{11}+21v^{13}+24v^{15}+24v^{17}\\
&+28v^{19}+6v^{21})z^3,\\
H(L_3,v,z)=&\,(3v^9+v^{15}-3v^{17}-v^{21})z^{-3}\\
&+(3v^{13}-9v^{17}+6v^{19})z^{-1}\\
&+(v^7+v^9+4v^{11}+7v^{13}-2v^{15}-11v^{17})z\\
&+(v^3+3v^5+6v^7+9v^9+12v^{11}+12v^{13}+6v^{15})z^3.
\end{aligned}
$$

(3) Chirality identification for the class of polyhedral links $D_A(\widehat{G^-})$.

We have

$$H(D_A(\widehat{G^-}),v,z)=z^{-3}P_1+z^{-1}P_2+zP_3+z^3P_4,$$

where P_2, P_3 and P_4 are the same as mentioned above. If $D_A(\widehat{G^-})$ is achiral, $H(D_A(\widehat{G^-}),v,z)$ must be symmetric in v. These polynomials P_1, P_2, P_3 and P_4 must be also symmetric in v.

Likewise, we have

$$P_1=(v^{2N-1}-v^{2N+1})(v^{-2}-2+v^2),\ N=n_1+\cdots+n_6\geq 6.$$

It is clear that the polynomial $v^{2N-1}-v^{2N+1}$ is not symmetric in v when $N\geq 6$. On the other hand, the polynomial $v^{-2}-2+v^2$ is symmetric in v, so P_1 must be not symmetric in v. Therefore such polyhedral links $D_A(\widehat{G^-})$ of this class are all chiral by the property (3) of the HOMFLY polynomial.

Similarly, we can explore the topological characters for the other three classes of polyhedral links $D_B(\widehat{G^-})$, $D_C(\widehat{G^+})$ and $D_D(\widehat{G^+})$.

4 Conclusions

In the paper, given an arbitrary polyhedron G, we generate several classes of polyhedral links $D_A(\widehat{G^-})$, $D_B(\widehat{G^-})$, $D_C(\widehat{G^+})$ and $D_D(\widehat{G^+})$ by applying the operation of '*X-tangle covering*' to the related reduced sets R(G^-) and R(G^+), where $D_A(\widehat{G^-})$ is the family of image-mirrors of $D_C(\widehat{G^+})$ and $D_B(\widehat{G^-})$ is of $D_D(\widehat{G^+})$. And then, we give the relationships between the W-polynomial of a polyhedral graph and the HOMFLY polynomials of polyhedral links which make it possible to establish a general formula for HOMFLY polynomial. This method simplifies the computation of the HOMFLY polynomial for polyhedral links, and thus facilitating the subsequent identification of the topological link type and chirality of polyhedral links. It may be promising that our method can be applied to the construction of more complicated polyhedral links and, the analyses of their topological properties.

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